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LETTER TO THE EDITOR

Non-linear dynamic systems, limit cycles, transformation groups, and perturbation techniques

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Abstract. A connection between non-linear systems of differential equations containing limit cycles, transformation groups, and perturbation techniques is discussed.

Recently, in a series of papers, the author (Steeb 1977a, b, c, d) has investigated a connection between Lie's theory of one-parameter groups and autonomous non-linear systems of differential equations ($\dot{x} = Y(x)$) containing periodic orbits. In particular, such systems have been considered where the periodic orbits are limit cycles.

For a given system of linear differential equations $\dot{x} = X(x) = Ax$ (where A is an $n \times n$ matrix) with periodic solutions, the non-linear system $\dot{x} = Y(x)$ is constructed via the relation

$$[X, Y] = \lambda Y \tag{1}$$

or, in a modern form,

$$L_X\alpha = \lambda\alpha. \tag{2}$$

[,] denotes the commutator of the C^∞-vector fields X and Y. The fields X and Y are written in local coordinates as $X = X_1\,\partial/\partial x_1 + \ldots + X_n\,\partial/\partial x_n$ and $Y = Y_1\,\partial/\partial x_1 + \ldots + Y_n\,\partial/\partial x_n$. $L_X\alpha$ stands for the Lie derivative of the differential form α with respect to X, where α is given by the inner product (contraction operation) $\alpha = Y \lrcorner\, \omega$ and $\omega(\omega = \mathrm{d}x_1 \times \ldots \times \mathrm{d}x_n)$ is the standard volume in $\mathbb{R}^n$. λ is a C^∞-function.

In some of the cited papers we have assumed that the vector field X is generated via the Hamiltonian form of the equations of motion. Note that the given approach also works if the equation $\dot{x} = X(x)$ is not generated by a Hamiltonian function.

Three problems arise. The first integrals of the subsidiary equations $\mathrm{d}x_1/X_1 = \ldots = \mathrm{d}x_n/X_n$ (relative invariants with respect to X) cannot be found, in general, when the vector field X is non-linear. Any dynamical system $\dot{x} = X(x)$ possesses $n-1$ (local) independent first integrals. Moreover, for a given X we cannot, in general, find a vector field Y (Y linear independent of X) such that equation (1) holds. In most of the examples (Steeb 1977a, b, c) the system of differential equations $\dot{x} = X(x)$ was linear. In this case we are able to find the most general vector field Y which commutes with X. Examples for the two-dimensional case, where the vector field X is non-linear, can be found in Steeb (1977d). Finally, assuming that the vector field Y is given (for example the vector field Y associated with the van der Pol equation $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$), then we cannot find an appropriate X explicitly.

To treat such problems, i.e. non-linear vector fields X or Y, we must include perturbation techniques in our approach. An appropriate perturbation technique is that developed by Kruskal (Kruskal 1962, McNamara and Whiteman 1967, Rae and Davidson 1973, Kummer 1971). Another technique, closely related to that of Kruskal, was developed by Moser (Moser 1966, 1967, Kummer 1971).

In the present Letter we discuss the connection between the perturbation technique of Kruskal and our algebraic approach for obtaining limit cycles.

Before considering the technique described by Kruskal we give a simple example (Steeb 1977c) which will serve to illustrate the connection between our approach and the perturbation technique. Let X be the vector field

$$X = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}. \tag{3}$$

Then the most general vector field Y which commutes with X has the form

$$Y = (x_1 f_2(r) - x_2 f_1(r)) \frac{\partial}{\partial x_1} + (x_2 f_2(r) + x_1 f_1(r)) \frac{\partial}{\partial x_2}. \tag{4}$$

To obtain the vector field Y we consider the Abelian Lie algebra $\{x_1\, \partial/\partial x_2 - x_2\, \partial/\partial x_1, x_1\, \partial/\partial x_1 + x_2\, \partial/\partial x_2\}$ and the rule $[X, fZ] = (Xf)Z + f[X, Z]$. As an abbreviation we have put $r^2 = x_1^2 + x_2^2$. In polar coordinates ($x_1 = r\cos\varphi$, $x_2 = -r\sin\varphi$, $0 < r < \infty$, $0 \leq \varphi < 2\pi$) the vector fields take the form

$$X = \frac{\partial}{\partial \varphi}, \qquad Y = f_2(r)\frac{\partial}{\partial r} + f_1(r)\frac{\partial}{\partial \varphi}. \tag{5}$$

Then the corresponding systems of differential equations become

$$\dot{\varphi} = 1; \qquad \dot{r} = 0 \tag{6}$$

$$\dot{\varphi} = f_1(r); \qquad \dot{r} = f_2(r). \tag{7}$$

The right-hand sides of both equations (7) do not depend on the angular variable φ. The necessary condition for the limit cycle, obtained via $X_1 Y_2 - Y_1 X_2 = 0$, leads to $f_2(r) = 0$.

As a concrete example, we set $f_2(r) = 1 - r^2$ and $f_1(r) = 1$. Then the only critical point of the system is $(x_1, x_2) = (0, 0)$ and the limit cycle is given by $r^2 = 1$. The limit cycle can also be viewed as a one-dimensional integral manifold of the system given above.

Kruskal (Kruskal 1962, McNamara and Whiteman 1967) has considered an autonomous system of differential equations $x = F(x, \epsilon)$ where it is assumed that for $\epsilon = 0$ the point $x(t)$ traces out closed curves as t increases. Then one can introduce new variables $y (y = (y_1, \ldots, y_{n-1}))$ and an angle-like variable φ to obtain the system

$$\dot{\varphi} = f(\varphi, y) \tag{8}$$

$$\dot{y} = \epsilon g(\varphi, y) \tag{9}$$

where f and g are periodic in φ with the period τ. The method developed by Kruskal

(1962) consists of introducing new variables ($z = Z(\varphi, y)$, $\phi = \Phi(\varphi, y)$) which lead to the equations

$$\dot{\phi} = \omega(z) \tag{10}$$

$$\dot{z} = \epsilon h(z). \tag{11}$$

The angular variable does not appear on the right-hand side of equations (10) and (11). Kruskal (1962) has shown that starting from φ, y it is possible to obtain new variables ϕ, z. The quantities Z, Φ, ω, and h are power series in ϵ. Moreover, he has also shown that it is possible to find the inverse transformation as a power series in ϵ.

Now let us study the equations (10) and (11) with $n = 2$. To the differential equations (10) and (11) we assign the vector field

$$Y = \omega(z)\frac{\partial}{\partial\varphi} + \epsilon h(z)\frac{\partial}{\partial z}. \tag{12}$$

According to the example described in the first part of the paper, the most general vector field X which commutes with Y has the form $X = \partial/\partial\varphi$. Then the necessary condition for the limit cycle leads to $h(z) = 0$. Since h and ω are infinite power series in ϵ, the vector field Y and therefore the equation for the limit cycle (necessary condition) can only be given approximately. We note that Moser (Moser 1966, 1967, Kummer 1971) described a method for obtaining the vector field Y in an approximate sense with a Lie algebraic method.

The problem can also be considered from the point of view of the dimension of the underlying Lie algebra. In our approach the vector field X is an element of a finite-dimensional Lie algebra (Steeb 1977a, b, c). For example, the vector field $X = (y, -x)$ is the basis element of the Lie algebra SO(2). On the other hand, the perturbation technique leads, in general, to infinite-dimensional Lie algebras (Sternberg 1961, Moser 1967).

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